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CIRCULAR MOTION OF BODIES OF REVOLUTION

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SUMMARY

This note extends the method of N.A.C.A. Report No. 516 to the case of an arbitrary body of revolution in steady motion at constant angle of attack and constant path curvature, viz, circular path. Expressions are given for the pressure, for the lateral and longitudinal components of the centrifugal force of the apparent mass, and for the yawing moment. These expressions are then applied to an ellipsoid of revolution mainly for the purpose of demonstrating the usefulness and simplicity of the vector method.

INTRODUCTION

The circular motion for airship-like bodies has thus far been calculated only for a prolate ellipsoid of revolution (reference 1, p. 133 and reference 2). In this paper, however, the circular motion of elongated bodies of revolution more nearly resembling airships will be investigated. The results will give the effect of rotation on the pressure distribution and thus yield some information as to the stresses set up in an airship in circular flight.

THE BOUNDARY CONDITIONS

The body is referred to a rectangular Cartesian frame OXYZ attached to it, and its axis of symmetry OX moves in a plane X'Y' fixed in space. The body is assumed to rotate uniformly in a counterclockwise manner about O'Z' with angular velocity Ω and to possess an angle of yaw β with respect to the origin O. (See fig. 1.)

When considering elongated bodies, it is convenient to introduce in any meridian plane XH confocal elliptic coordinates μ, λ. The coordinates x, y, z, of any point P in space are then given in this new orthogonal system of coordinates by the following expressions:

$$\left.\begin{aligned} x &= 2a\lambda\mu \\ y &= 2a\,(\lambda^2 - 1)^{\frac{1}{2}}\,(1 - \mu^2)^{\frac{1}{2}}\cos\omega \\ z &= 2a\,(\lambda^2 - 1)^{\frac{1}{2}}\,(1 - \mu^2)^{\frac{1}{2}}\sin\omega \end{aligned}\right\} \qquad (1)$$

The coordinate surfaces of the so-called "spheroidal coordinates" μ, λ, ω are obtained by setting, in turn, μ constant (hyperboloids of two sheets), λ constant (prolate ellipsoids of revolution), ω constant (half planes through the axis of symmetry OX); these coordinate surfaces furnish by their intersections the three coordinate lines: μ variable (ellipses), λ variable (hyperbolas), and ω variable (circles with centers on the OX axis). The coordinate lines give by means of their tangents, directed positively in the direction in which the corresponding coordinate is increasing, a rectangular Cartesian frame whose origin is the point in which the three coordinate surfaces (or coordinate lines) intersect. In order that this frame be a right-handed one it is so arranged that the positive direction of the tangents to the μ, λ, ω coordinate lines are analogous, respectively, to OX, OY, and OZ.

The position of the moving axes is defined by the position of the origin O, whose coordinates with respect to the fixed axes are $R\cos\Omega t$, $R\sin\Omega t$, 0, and by the direction cosines of one set of axes with respect to the other. The following direction cosine table gives these values at any time t:

	x	y	z
x'	$\sin(\Omega t - \beta)$	$\cos(\Omega t - \beta)$	0
y'	$-\cos(\Omega t - \beta)$	$\sin(\Omega t - \beta)$	0
z'	0	0	1

The velocity of any point fixed with regard to the moving axes is the resultant of two vectors, one of which $\overline{V}$ is the same for all points of the system, being independent of the coordinates x, y, z of the point and having the components in the directions O'X', O'Y', O'Z' equal to $-R\Omega \sin \Omega t$, $R\Omega \cos \Omega t$, 0, respectively, and in the directions OX, OY, OZ equal to

$$\left.\begin{aligned} V_x &= -R\Omega \cos \beta \\ V_y &= -R\Omega \sin \beta \\ V_z &= 0 \end{aligned}\right\} \qquad (2)$$

This part of the motion is therefore a translation and is made up of an axial motion with velocity $-R\Omega \cos \beta$ and a transverse motion with velocity $-R\Omega \sin \beta$. The other part of the motion has components with respect to the instantaneous position of the OXYZ frame, given by

$$\left.\begin{aligned} t_x &= -\Omega y \\ t_y &= \Omega x \\ t_z &= 0 \end{aligned}\right\} \qquad (3)$$

and is therefore the vector product of a vector $\bar{\Omega}$ whose components with regard to the moving axes are 0, 0, Ω, and of the radius vector $\bar{r}$ from the origin O to the point P. The vector $\bar{t}$ lies in the XY plane and is perpendicular to both $\bar{\Omega}$ and $\bar{r}$ and has a magnitude equal to $\Omega r \sin (\Omega r)$. It accordingly represents a motion due to a rotation of the body with angular velocity Ω about the OZ axis. Referred to a meridian plane making an angle ω with the XY plane, the vector $\bar{t}$ has a component $t_x = -\Omega h \cos \omega$ in the direction of the X axis, a component $t_h = \Omega x \cos \omega$ in the direction of the H axis, and a component $\Omega x \sin \omega$ perpendicular to the meridian plane. This latter component plays no part in fixing the boundary conditions for a body of revolution.

The translational velocity V of the body gives rise to motions in the fluid represented by two velocity potentials: φ_1, due to the axial velocity V_x and φ_2, due to the transverse velocity V_y. This type of motion has already been discussed in detail in reference 3. Disregarding for the present the translational velocity V, the normal component of the velocity at any point P of the meridian profile, whose element of length is ds, is given by

$$v_n = \Omega \cos \omega \left(h \frac{dh}{ds} + x \frac{dx}{ds}\right) \qquad (4)$$

Furthermore, if φ_3 represents the velocity potential of the flow due to the rotational part of the motion, the normal component of the velocity with respect to the Cartesian (μ, λ, ω) frame attached to the point P is

$$-\frac{\partial\varphi_3}{\partial n} = \frac{\partial\varphi_3}{\partial s_\mu}\frac{ds_\lambda}{ds} - \frac{\partial\varphi_3}{\partial s_\lambda}\frac{ds_\mu}{ds}$$

or

$$\frac{\partial\varphi_3}{\partial s_\mu} ds_\lambda - \frac{\partial\varphi_3}{\partial s_\lambda} ds_\mu = \Omega \cos\omega \; (hdh + xdx) \qquad (5)$$

where ds_λ, ds_μ are the linear elements corresponding to the λ, μ coordinate lines, respectively.

Since $x = 2a\lambda\mu$, $h = 2a\,(\lambda^2 - 1)^{\frac{1}{2}}\,(1 - \mu^2)^{\frac{1}{2}}$

and $ds_\mu = 2a\left(\dfrac{\lambda^2 - \mu^2}{1 - \mu^2}\right)^{\frac{1}{2}} d\mu$, $ds_\lambda = 2a\left(\dfrac{\lambda^2 - \mu^2}{\lambda^2 - 1}\right)^{\frac{1}{2}} d\lambda$

it follows that

$$\frac{(1-\mu^2)\dfrac{\partial\varphi_3}{\partial\mu}\,d\lambda - (\lambda^2-1)\dfrac{\partial\varphi_3}{\partial\lambda}\,d\mu}{(\lambda^2 - 1)^{\frac{1}{2}}\,(1-\mu^2)^{\frac{1}{2}}} = (2a)^2\,\Omega\,\cos\omega\,(\mu d\mu + \lambda d\lambda) \qquad (6)$$

The velocity potential φ_3 satisfies Laplace's equation and may, in general, be represented by the following expansion:

$$\sum_m \sum_n C_n^m\,P_n^m(\mu)\,Q_n^m(\lambda)\,\cos m\omega + \sum_m \sum_n D_n^m\,P_n^m(\mu)\,Q_n^m(\lambda)\sin m\omega$$

where $P_n^m(\mu)$, $Q_n^m(\lambda)$ denote the associated Legendre functions of degree n and order m of the first and second kind, respectively; and C_n^m, D_n^m signify certain constants which are to be determined by the boundary conditions. From the form of the boundary condition (6) it is clear that

$$\varphi_3 = \sum_{n=1}^{\infty} C_n^1\,P_n^1\,(\mu)\,Q_n^1\,(\lambda)\,\cos\omega \qquad (7)$$

The functions $P_n^1(\mu)$, $Q_n^1(\lambda)$ are given in terms of the corresponding Legendre polynomials by means of the following expressions:

$$\left.\begin{aligned} P_n^1(\mu) &= (1-\mu^2)^{\frac{1}{2}} \frac{dP_n(\mu)}{d\mu} \\ Q_n^1(\lambda) &= (\lambda^2-1)^{\frac{1}{2}} \frac{dQ_n(\lambda)}{d\lambda} \end{aligned}\right\} \tag{8}$$

where $P_n(\mu)$, $Q_n(\lambda)$ satisfy Legendre's differential equation. With the use of equations (7) and (8) together with Legendre's differential equation for $P_n(\mu)$ and $Q_n(\lambda)$, the boundary condition (6) takes the following form:

$$\sum_{n=1}^{\infty} C_n^1 \left[\frac{d(\lambda\mu)}{d\mu} \frac{dP_n}{d\mu} \frac{dQ_n}{d\lambda} - n(n+1) \frac{d}{d\mu}(P_n Q_n) \right] = (2a)^2 \Omega \left(\mu + \frac{1}{2} \frac{d\lambda^2}{d\mu}\right) \tag{9}$$

where it is assumed that $\lambda = \lambda(\mu)$ describes the meridian curve of the body in elliptic coordinates.

For example, if the body is a prolate ellipsoid of revolution its meridian curve is given simply by $\lambda = \lambda_o$, a constant. Equation (9) then becomes:

$$\sum_{n=1}^{\infty} C_n^1 \left[\lambda_o \left(\frac{dQ_n}{d\lambda}\right)_{\lambda=\lambda_o} - n(n+1)\, Q_n(\lambda_o)\right] \frac{dP_n}{d\mu} = (2a)^2 \Omega \mu$$

and this expression must be valid for the entire range of μ. Now, of the complete set of P_n's only one of them, namely, $P_2 = \frac{1}{2}(3\mu^2 - 1)$, has its derivative proportional to μ. Hence, $n = 2$ is the only term appearing in the boundary condition for the ellipsoid. Therefore (reference 1, p. 133),

$$3C_2^1 = -\frac{(2a)^2 \Omega}{\frac{3}{2}(2\lambda_o^2 - 1)\log\frac{\lambda_o+1}{\lambda_o-1} - 6\lambda_o + \frac{\lambda_o}{\lambda_o^2-1}} = C$$

and

$$\varphi_3 = C\mu(1-\mu^2)^{\frac{1}{2}} (\lambda^2-1)^{\frac{1}{2}} \left(\frac{3}{2}\lambda \log\frac{\lambda+1}{\lambda-1} - 3 - \frac{1}{\lambda^2-1}\right) \cos\omega$$

Thus, the complete velocity potential consists of a part φ_1 due to an axial velocity $-R\Omega\cos\beta$, a part φ_2 due to a transverse velocity $-R\Omega\sin\beta$, and a part φ_3 due to a rotational motion of angular velocity Ω about OZ. It follows then that

$$\varphi = \varphi_1 + \varphi_2 + \varphi_3 \tag{10}$$

where

$$\varphi_1 = \sum_{n=1}^{\infty} A_n P_n(\mu)\, Q_n(\lambda) \quad \text{with} \quad \sum_{n=1}^{\infty} \frac{A_n}{n(n+1)} \frac{dP_n}{d\mu} \frac{dQ_n}{d\lambda} = aR\Omega\cos\beta \quad \text{as the boundary condition}$$

$$\varphi_2 = \sum_{n=1}^{\infty} B_n^1 P_n^1(\mu)\, Q_n^1(\lambda) \cos\omega \quad \text{with} \quad \sum_{n=1}^{\infty} B_n^1 \left[\frac{d(\lambda\mu)}{d\mu} \frac{dP_n}{d\mu} \frac{dQ_n}{d\lambda} - n(n+1) \frac{d}{d\mu}(P_n Q_n)\right] =$$

$$-2aR\Omega\sin\beta\left(\lambda + \mu\frac{d\lambda}{d\mu}\right) \text{ as the boundary condition (reference 3)}$$

and

$$\varphi_3 = \sum_{n=1}^{\infty} C_n^1 P_n^1(\mu)\, Q_n^1(\lambda) \cos\omega \quad \text{with} \quad \sum_{n=1}^{\infty} C_n^1 \left[\frac{d(\lambda\mu)}{d\mu} \frac{dP_n}{d\mu} \frac{dQ_n}{d\lambda} - n(n+1) \frac{d}{d\mu}(P_n Q_n)\right] =$$

$$(2a)^2\,\Omega\left(\mu + \lambda\frac{d\lambda}{d\mu}\right) \text{ as the boundary condition}$$

From the similarity of the left-hand sides of the boundary conditions associated with φ_2, φ_3, it is clear that the treatment for φ_2 applies equally well to φ_3. Thus, a method has been outlined for obtaining the velocity potential in a fluid due to a body of revolution in circular flight.

THE PRESSURE FORMULA

The next step is to calculate the forces acting on the moving body owing to the pressure of the fluid medium. Since the body moves at a constant angle of attack and in a circular path with constant angular velocity, the velocities $\bar{V}$, $\bar{t}$ are independent of the time so that the motion is a steady one. Therefore, the velocity potential φ does not contain the time t explicitly, i.e., $\frac{\partial\varphi}{\partial t} = 0$, and Lamb's formula (reference 1, p. 18) for the pressure when the coordinate axes are in motion becomes:

$$\frac{p}{\rho} = -\frac{1}{2}\,q^2 - \omega_x\left(y\frac{\partial\varphi}{\partial x} - z\frac{\partial\varphi}{\partial y}\right) - \omega_y\left(z\frac{\partial\varphi}{\partial x} - x\frac{\partial\varphi}{\partial z}\right) - \omega_z\left(x\frac{\partial\varphi}{\partial y} - y\frac{\partial\varphi}{\partial x}\right) \quad (11)$$

where ω_x, ω_y, ω_z are the component rotations referred to the moving axes and $q^2 = (u-U)^2 + (v-V)^2 + (w-W)^2$ where $(u,v,w) = \left(-\frac{\partial\varphi}{\partial x}, -\frac{\partial\varphi}{\partial y}, -\frac{\partial\varphi}{\partial z}\right)$; also $\varphi = \varphi_1 + \varphi_2 + \varphi_3$.

If the radius vector to the point (x,y,z) is denoted by $\bar{r}$, the fluid-velocity vector by $-\overline{\mathrm{grad}}\,\varphi$, and the rotation by $\bar{\Omega}$ then equation (11) may be written in vector notation as:

$$\frac{p}{\rho} = -\frac{1}{2}\,(\bar{q}\cdot\bar{q}) - \left(\bar{\Omega}\cdot[\bar{r}\times\overline{\mathrm{grad}}\,\varphi]\right) \quad (12)$$

where the symbols ( ) and [ ] denote, respectively, the scalar and vector products of two vectors. The scalar triple product $(\bar{\Omega}\cdot[\bar{r}\cdot\overline{\mathrm{grad}}\,\varphi]) = ([\bar{\Omega}\cdot\bar{r}]\cdot\overline{\mathrm{grad}}\,\varphi) = (\bar{t}\cdot\overline{\mathrm{grad}}\,\varphi)$ where $\bar{t}$ denotes that part of the velocity vector of any point attached to the moving axes due to the rotation alone. Equation (12), applied to the problem of this paper, then becomes:

$$\frac{p}{\rho} = -\frac{1}{2}\,(V_x^2 + V_y^2) - \frac{1}{2}\,(\overline{\mathrm{grad}}\,\varphi\cdot\overline{\mathrm{grad}}\,\varphi) - (\bar{V} + \bar{t}\cdot\overline{\mathrm{grad}}\,\varphi)$$

or, omitting $\frac{1}{2}\,(V_x^2 + V_y^2)$ since it is a constant and therefore has the same value over the surface of the body, it follows that:

$$\frac{p}{\rho} = \frac{1}{2}\,(\bar{V} + \bar{t}\cdot\bar{V} + \bar{t}) - \frac{1}{2}\,(\overline{\mathrm{grad}}\,\varphi + \bar{V} + \bar{t}\cdot\overline{\mathrm{grad}}\,\varphi + \bar{V} + \bar{t}) \quad (13)$$

In order to interpret equation (13) in spheroidal coordinates, it is necessary to obtain the presentation of the various vectors involved, in the Cartesian frame attached to the point (μ, λ, ω). The table of direction cosines for the Cartesian frame attached to any point is:

	x	y	z	
μ	$\lambda\left(\frac{1-\mu^2}{\lambda^2-\mu^2}\right)^{\frac{1}{2}}$	$-\mu\left(\frac{\lambda^2-1}{\lambda^2-\mu^2}\right)^{\frac{1}{2}}\cos\omega$	$-\mu\left(\frac{\lambda^2-1}{\lambda^2-\mu^2}\right)^{\frac{1}{2}}\sin\omega$	
λ	$\mu\left(\frac{\lambda^2-1}{\lambda^2-\mu^2}\right)^{\frac{1}{2}}$	$\lambda\left(\frac{1-\mu^2}{\lambda^2-\mu^2}\right)^{\frac{1}{2}}\cos\omega$	$\lambda\left(\frac{1-\mu^2}{\lambda^2-\mu^2}\right)^{\frac{1}{2}}\sin\omega$	(14)
ω	0	$-\sin\omega$	$\cos\omega$	

With the use of this table of direction cosines the vector $\overline{q}$, which for the present problem has the components $-\frac{\partial\varphi}{\partial x} - V_x$, $-\frac{\partial\varphi}{\partial y} - V_y$, $-\frac{\partial\varphi}{\partial z}$ with respect to the instantaneous position of the moving axes, has in the Cartesian frame attached to any point μ, λ, ω the following components:

$$\left.\begin{aligned} q_\mu &= -\frac{\partial\varphi}{\partial s_\mu} - V_\mu \\ q_\lambda &= -\frac{\partial\varphi}{\partial s_\lambda} - V_\lambda \\ q_\omega &= -\frac{\partial\varphi}{\partial s_\omega} - V_\omega \end{aligned}\right\} \qquad (15)$$

where

$$\frac{\partial\varphi}{\partial s_\mu} = \frac{1}{2a}\left(\frac{1-\mu^2}{\lambda^2-\mu^2}\right)^{\frac{1}{2}}\frac{\partial\varphi}{\partial\mu}, \quad \frac{\partial\varphi}{\partial s_\lambda} = \frac{1}{2a}\left(\frac{\lambda^2-1}{\lambda^2-\mu^2}\right)^{\frac{1}{2}}\frac{\partial\varphi}{\partial\lambda},$$

$$\frac{\partial\varphi}{\partial s_\omega} = \frac{1}{2a\,(\lambda^2-1)^{\frac{1}{2}}\,(1-\mu^2)^{\frac{1}{2}}}\frac{\partial\varphi}{\partial\omega}$$

and

$$V_\mu = V_x \lambda \left(\frac{1-\mu^2}{\lambda^2-\mu^2}\right)^{\frac{1}{2}} - V_y \mu \left(\frac{\lambda^2-1}{\lambda^2-\mu^2}\right)^{\frac{1}{2}} \cos\omega,$$

$$V_\lambda = V_x \mu \left(\frac{\lambda^2-1}{\lambda^2-\mu^2}\right)^{\frac{1}{2}} + V_y \lambda \left(\frac{1-\mu^2}{\lambda^2-\mu^2}\right)^{\frac{1}{2}} \cos\omega, \quad V_\omega = - V_y \sin\omega$$

Similarly, the vector $\overline{\Omega}$, which in the moving OXYZ frame has the components $(0, 0, \Omega)$, has in the (μ, λ, ω) Cartesian frame the following presentation:

$$\left.\begin{aligned} \Omega_\mu &= - \Omega\mu \left(\frac{\lambda^2-1}{\lambda^2-\mu^2}\right)^{\frac{1}{2}} \sin\omega \\ \Omega_\lambda &= \Omega\lambda \left(\frac{1-\mu^2}{\lambda^2-\mu^2}\right)^{\frac{1}{2}} \sin\omega \\ \Omega_\omega &= \Omega \cos\omega \end{aligned}\right\} \tag{16}$$

and the radius vector $\overline{r}$, the following components:

$$\left.\begin{aligned} r_\mu &= 2a\mu \left(\frac{1-\mu^2}{\lambda^2-\mu^2}\right)^{\frac{1}{2}} \\ r_\lambda &= 2a\lambda \left(\frac{\lambda^2-1}{\lambda^2-\mu^2}\right)^{\frac{1}{2}} \\ r_\omega &= 0 \end{aligned}\right\} \tag{17}$$

The vector $\overline{t}$ has the components $-\Omega y, \Omega x, 0$ along the axes of the moving frame and the following components relative to the rectangular (μ, λ, ω) axes:

$$\left.\begin{aligned} t_\mu &= - 2a\,\Omega\lambda \left(\frac{\lambda^2-1}{\lambda^2-\mu^2}\right)^{\frac{1}{2}} \cos\omega \\ t_\lambda &= 2a\,\Omega\mu \left(\frac{1-\mu^2}{\lambda^2-\mu^2}\right)^{\frac{1}{2}} \cos\omega \\ t_\omega &= - 2a\,\Omega\lambda\mu \sin\omega \end{aligned}\right\} \tag{18}$$

THE LONGITUDINAL COMPONENT OF THE CENTRIFUGAL FORCE OF THE APPARENT MASS

If θ is the inclination to the X axis of the outward-drawn normal to the meridian curve at the point P, then the longitudinal force is

$$F_L = -\int_{s_1}^{s_2}\int_0^{2\pi} p \cos\theta \, h d\omega ds$$

where ds is an element of length along a meridian curve; s_1, s_2 the values of s at the fore and aft ends of the body, and h the radius of the section through P perpendicular to the axis of symmetry OX.

Putting

$$-\cos\theta = \frac{dh}{ds}$$

it follows that:

$$F_L = \frac{1}{2}\int_c\int_0^{2\pi} p d\omega dh^2$$

The area of the section of radius h is $A = \pi h^2$ and the average pressure at this section is defined to be

$$p' = \frac{1}{2\pi}\int_0^{2\pi} p d\omega$$

It then follows that

$$F_L = \int_c p' dA \tag{19}$$

taken over the generating contour of the body of revolution.

THE TRANSVERSE COMPONENT OF THE CENTRIFUGAL FORCE OF THE APPARENT MASS

The component of the pressure p in the plane XY is $p \cos \omega$ and the resultant transverse force is therefore

$$F_T = -\int_{s_1}^{s_2}\int_0^{2\pi} p \cos \omega \sin \theta \, h d\omega \, ds$$

or, since $\sin \theta = \frac{dx}{ds}$

$$F_T = -\int_{x_1}^{x_2}\int_0^{2\pi} hp \cos \omega \, d\omega dx$$

The resultant of the components of the pressure p in the XY plane taken round a section of radius h perpendicular to the X axis, is given by

$$P'' = h \int_0^{2\pi} p \cos \omega \, d\omega$$

acting normally to the meridian curve in the XY plane. Then

$$F_T = -\int_{x_1}^{x_2} P'' \, dx \tag{20}$$

THE YAWING MOMENT

The yawing moment about the axis of Z is given by

$$M_z = \int_{s_1}^{s_2} P'' l \, ds$$

where l is the perpendicular distance from the origin O to the line of action of P'' (or the normal at the point of the meridian curve in the XY plane at which P'' is applied). The direction cosines at a point P (x,y), of the outward-drawn normal to the meridian curve in the XY plane are

$$-\frac{dy}{ds}, \frac{dx}{ds}$$

It follows, by means of elementary analytic geometry, that

$$l = - \frac{xdx + ydy}{ds} = - r \frac{dr}{ds}$$

and hence

$$M_z = - \frac{1}{2} \int_c P'' \, d\,(x^2 + y^2) = - \frac{1}{2} \int_c P'' \, dr^2 \qquad (21)$$

taken over the upper half of the meridian profile in the XY plane. It is interesting to note that for a sphere, since r^2 = constant, $M_z = 0$. The differential dr^2 may be looked upon as a measure of the deviation of the body from a sphere.

Langley Memorial Aeronautical Laboratory,
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APPENDIX

SOLUTION FOR AN ELLIPSOID

The case of an ellipsoid of revolution moving in a circular path has already been worked out, but the methods used are very cumbersome and involve unnecessary algebraic manipulations. It seems therefore instructive of the present method, which makes liberal use of vector analysis, to redevelop the solution for an ellipsoid.

The equation of the meridian ellipse in Cartesian coordinates is:

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1$$

and in ellipsoidal coordinates, simply

$$\lambda = \lambda_0$$

where

$$\lambda_0 = \frac{1}{e}, \quad A^2 - B^2 = (2a)^2, \quad Ae = (2a), \quad \text{and } 2a\,(\lambda_0^2 - 1)^{\frac{1}{2}} = B$$

The velocity potential φ is the sum of the following three components (reference 1, pp. 132 and 133):

$$
\left.
\begin{aligned}
\varphi_1 &= \frac{-Au}{\frac{1}{2e}\log\frac{1+e}{1-e} - \frac{1}{1-e^2}}\left(\frac{1}{2}\lambda\log\frac{\lambda+1}{\lambda-1} - 1\right)\mu \\[2ex]
\varphi_2 &= \frac{-Aev}{\frac{1}{2}\log\frac{1+e}{1-e} - \frac{e-2e^3}{1-e^2}}\,(\lambda^2-1)^{\frac{1}{2}}\,(1-\mu^2)^{\frac{1}{2}}\left(\frac{1}{2}\log\frac{\lambda+1}{\lambda-1} - \frac{\lambda}{\lambda^2-1}\right)\cos\omega \\[2ex]
\varphi_3 &= \frac{-(A^2-B^2)\,\Omega}{\frac{3}{2}\,\frac{2-e^2}{e^3}\log\frac{1+e}{1-e} - \frac{6}{e} + \frac{e}{1-e^2}}\,\mu(\lambda^2-1)^{\frac{1}{2}}\,(1-\mu^2)^{\frac{1}{2}}\left(\frac{3}{2}\lambda\log\frac{\lambda+1}{\lambda+1} - 3 - \frac{1}{\lambda^2-1}\right)\cos\omega
\end{aligned}
\right\}\quad(22)
$$

where

$$u = -V\cos\beta \quad\text{and}\quad v = -V\sin\beta$$

The pressure formula (equation (13)) is expressed in vector form and is therefore independent of the particular coordinate system used. The components of the vectors $\overline{\text{grad}}\ \varphi$, $\overline{V}$, and $\overline{t}$ are given by equations (16) to (18). For the simple case of an elliptical profile they are as follows:

$$\overline{\mathrm{grad}}\,\varphi+\overline{V}+\overline{t} \equiv \begin{cases} u\left(\dfrac{1-\mu^2}{1-e^2\mu^2}\right)^{\frac{1}{2}}(1+L)-v\mu\left(\dfrac{1-e^2}{1-e^2\mu^2}\right)^{\frac{1}{2}}(1+M)\cos\omega-A\,\Omega\left(\dfrac{1-e^2}{1-e^2\mu^2}\right)^{\frac{1}{2}}\left\{1+(2\mu^2-1)e^2N\right\}\cos\omega \\ 0 \\ -\left\{v(1+M)+A\,\Omega\mu\,(1+e^2N)\right\}\sin\omega \end{cases} \tag{23}$$

$$\overline{V}+\overline{t} \equiv \begin{cases} u\left(\dfrac{1-\mu^2}{1-e^2\mu^2}\right)^{\frac{1}{2}}-\left(\dfrac{1-e^2}{1-e^2\mu^2}\right)^{\frac{1}{2}}(v\mu+A\Omega)\cos\omega \\ u\mu\left(\dfrac{1-e^2}{1-e^2\mu^2}\right)^{\frac{1}{2}}+\left(\dfrac{1-\mu^2}{1-e^2\mu^2}\right)^{\frac{1}{2}}(v+2ae\mu\Omega)\cos\omega \\ -(v+A\mu\Omega)\sin\omega \end{cases} \tag{24}$$

where

$$\frac{\frac{1}{2e}\log\frac{1+e}{1-e}-1}{\frac{1}{2e}\log\frac{1+e}{1-e}-\frac{1}{1-e^2}}=-L,\quad \frac{\frac{1}{2}\log\frac{1+e}{1-e}-\frac{e}{1-e^2}}{\frac{1}{2}\log\frac{1+e}{1-e}-\frac{e-2e^3}{1-e^2}}=-M,\quad \frac{\frac{3}{2e}\log\frac{1+e}{1-e}-3-\frac{e^2}{1-e^2}}{\frac{3}{2}\frac{2-e^2}{e}\log\frac{1+e}{1-e}-6-\frac{e^2}{1-e^2}}=-N$$

$\frac{P}{\rho}=\frac{1}{2}$

Calculation of the Longitudinal Force Component F_L

According to equation (19)

$$F_L = \int_c P' \, dA = \pi \int_c P' \, dh^2 = - 2\pi B^2 \int_{-1}^{1} P' \mu \, d\mu$$

where

$$P' = \frac{1}{2\pi} \int_0^{2\pi} p \, d\omega$$

The pressure p is easily calculated by forming the scalar products of equation (13) by means of the vector components of equations (23) and (24). It is useful to observe that only the coefficients of $\cos^2 \omega$, $\sin^2 \omega$ need be considered since $\int_0^{2\pi} \cos^2 \omega d\omega = \int_0^{2\pi} \sin^2 \omega d\omega = \pi$ while $\int_0^{2\pi} \cos \omega d\omega = \int_0^{2\pi} \sin \omega d\omega = 0$. It then follows after some algebraic reductions, that

$$F_L = \frac{4}{3} \pi A B^2 \rho \Omega M v = - m M \frac{V^2}{R} \sin \beta$$

where $m = \frac{4}{3} \pi A B^2 \rho$, the mass of the fluid displaced by the spheroid. Noting that $M = k_T$, the transverse inertia coefficient of the body:

$$F_L = - m k_T \frac{V^2}{R} \sin \beta \tag{25}$$

Calculation of the Transverse Force Component F_T

According to equation (20)

$$F_T = - \int_c P'' \, dx = - A \int_{-1}^{1} P'' \, d\mu$$

where

$$P'' = h \int_0^{2\pi} p \cos \omega d\omega$$

Again, observing that only the coefficients of $\cos \omega$ in the expression for the pressure p need be considered, it follows that:

$$F_T = -\frac{4}{3}\pi A B^2 \rho \Omega L u = m L \frac{V^2}{R} \cos \beta$$

Noting that $L = k_a$, the axial inertia coefficient of the body:

$$F_T = m k_a \frac{V^2}{R} \cos \beta \tag{26}$$

Calculation of the Yawing Moment M_z.

According to equation (21)

$$M_z = -\frac{1}{2}\int_c P'' \, d(x^2 + y^2)$$

Now

$$x = 2a \lambda \mu \quad \text{and} \quad y = 2a(\lambda_o^2 - 1)^{\frac{1}{2}} (1 - \mu^2)^{\frac{1}{2}}$$

so that

$$M_z = -(2a)^2 \int_{-1}^{1} P'' \mu \, d\mu$$

Again only the coefficients of $\cos \omega$ in the expression for the pressure p need be considered and it follows that:

$$M_z = -\frac{\pi(2a)^2 B^2}{A} u v \rho (1+L)(1+M)\left[\frac{2(e^2-1)}{e^5}\left(\frac{1}{2}\log\frac{1+e}{1-e} + \frac{e-2e^3}{e^2-1}\right) + \frac{8}{3e^2}\right]$$

From the expressions for L, M it is easy to see that

$$\frac{1}{2(L+1)} + \frac{1}{(M+1)} = 1$$

or

$$\frac{M+1}{2} + (1+L) = (L+1)(M+1)$$

It may be verified that

$$\frac{4}{3}\,\frac{2M-1}{M+1} = \frac{2(e^2-1)}{e^3}\left(\frac{1}{2}\log\frac{1+e}{1-e} + \frac{e-2e^3}{e^2-1}\right) + \frac{8}{3}$$

It follows that

$$M_z = -m u v (1 + L)(2M - 1)$$

and since

$$(1 + L)\ (2M - 1) = M - L = k_T - k_a,$$

that

$$M_z = -\frac{mV^2}{2}\ (k_T - k_a)\ \sin 2\beta \tag{27}$$

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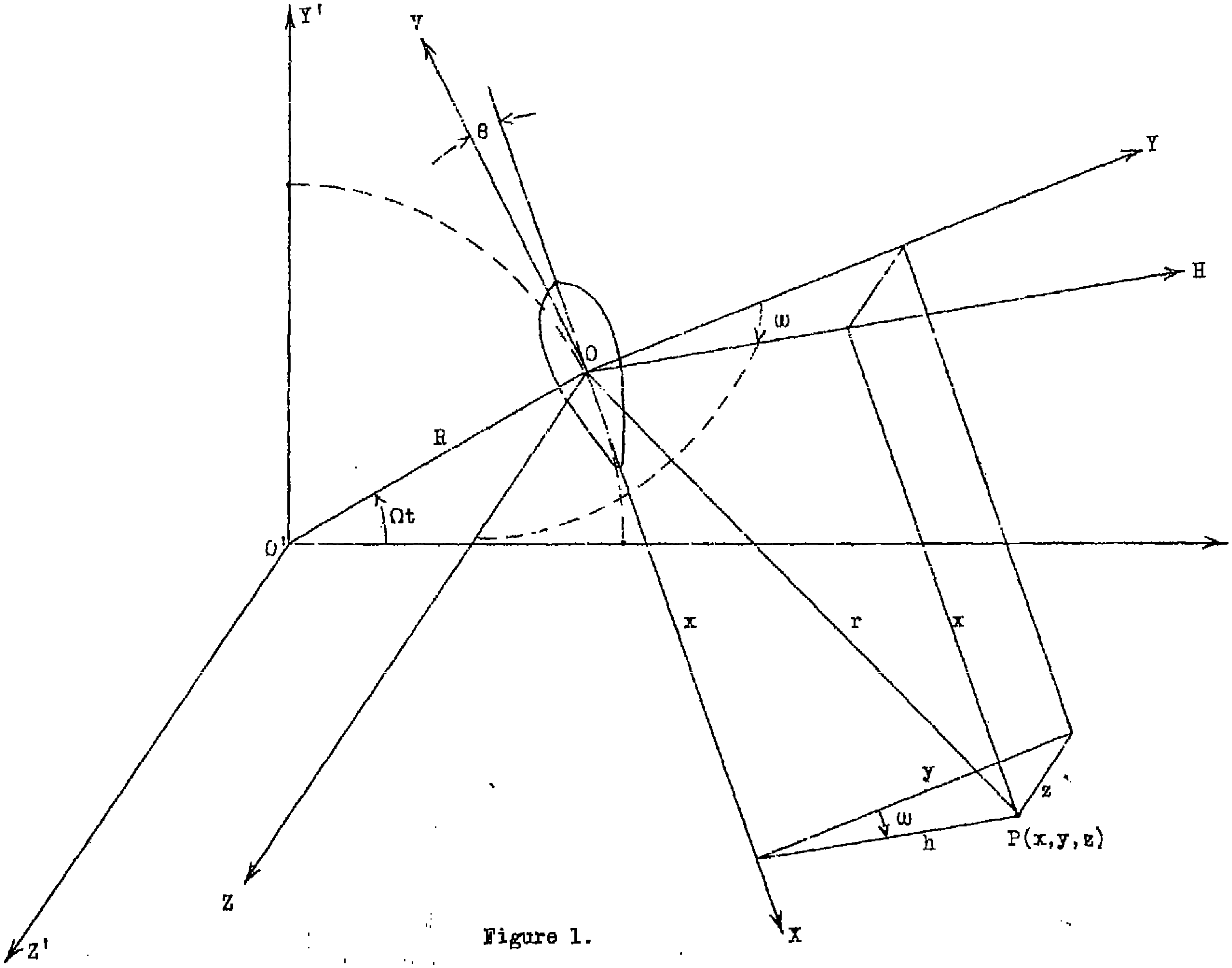

Figure 1.